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Symmetry problem

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Abstract

A novel approach to an old symmetry problem is developed. A proof is given for the following symmetry problem, studied earlier: if $\Delta u = 1$ in $D \subset \mathbb{R}^3$, $u = 0$ on S , the boundary of D , and $u_N = \text{const}$ on S , then S is a sphere. It is assumed that S is a Lipschitz surface homeomorphic to a sphere. This result has been proved in different ways by various authors. Our proof is based on a simple new idea.

MSC: 35J05, 31B20

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1 Introduction

Symmetry problems are of interest both theoretically and in applications. A well-known, and still unsolved, symmetry problem is the Pompeiu problem (see [9], [10], and the references therein). In modern formulation this problem consists of proving the following conjecture:

If $D \subset \mathbb{R}^n$, $n \geq 2$, is a domain homeomorphic to a ball, and the boundary S of D is smooth ($S \in C^{1,\lambda}$, $\lambda > 0$, is sufficient), and if the problem

$$(1) \quad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \quad u|_S = c \neq 0, \quad u_N|_S = 0, \quad k^2 = \text{const} > 0,$$

where c is a constant, has a solution, then S is a sphere.

A similar problem (*Schiffer's conjecture*) is also unsolved (see also [4]):

If the problem

$$(2) \quad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \quad u|_S = 0, \quad u_N|_S = c \neq 0, \quad k^2 = \text{const} > 0$$

has a solution, then S is a sphere.

Standing assumptions:

In this paper we assume that $D \subset \mathbb{R}^3$ is a bounded domain homeomorphic to a ball, with a sufficiently smooth boundary S (S is Lipschitz suffices).

We use the following notation: $D' = \mathbb{R}^3 \setminus D$, $B_R = \{x : |x| \leq R\}$, $B_R \supset D$, \mathcal{H} is the set of all harmonic functions in B_R , $R > 0$ is an arbitrary large number, such that the ball B_R contains D , $|D|$ and $|S|$ are the volume of D and the surface area of S , respectively.

In [12] it is proved that if

$$(3) \quad \int_D \frac{dy}{4\pi|x-y|} = \frac{c}{|x|}, \quad \forall x \in B'_R, \quad c = \text{const} > 0,$$

then D is a ball. The proof in [12] is based on an idea similar to the one we are using in this paper.

In [13] a symmetry problem of interest in elasticity theory is studied by A.D.Alexandrov's method of a moving plane ([1]), used also in [14]. The result in [14], which is formulated below in Theorem 1, was proved in [15] by a method, different from the one given in [14], and discussed also in [2]. The argument in [2] remained unclear to the author.

In [5] another symmetry problem of potential theory was studied.

Our goal is to give a new proof of Theorem 1. The result of Theorem 1 was obtained in [14] for \mathbb{R}^n , $n \geq 2$.

Theorem 1. *If $D \subset \mathbb{R}^3$ is a bounded domain, homeomorphic to a ball, S is its Lipschitz boundary, and the problem:*

$$(4) \quad \Delta u = 1 \quad \text{in } D, \quad u|_S = 0, \quad u_N|_S = c := \frac{|D|}{|S|} > 0$$

has a solution, then S is a sphere.

This result is equivalent to the following result:

If

$$(5) \quad \int_D h(x)dx = c \int_S h(s)ds, \quad \forall h \in \mathcal{H}, \quad c := \frac{|D|}{|S|},$$

then S is a sphere.

The equivalence of (4) and (5) can be proved as follows.

Suppose (4) holds. Multiply (4) by an arbitrary $h \in \mathcal{H}$, integrate by parts and get

$$(6) \quad \int_D h(x)dx = c \int_S h(s)ds.$$

If $h = 1$ in (6), then one gets $c = \frac{|D|}{|S|}$, so (6) is identical to (5).

Suppose (5) holds. Then (6) holds. Let v solve the problem $\Delta v = 1$ in D , $v|_S = 0$. This v exists and is unique. Using (6), the equation $\Delta h = 0$ in D , and the Green's formula, one gets

$$(7) \quad c \int_S h(s)ds = \int_D h(x)dx = \int_D h(x)\Delta v dx = \int_S h(s)v_N ds.$$

Thus,

$$(8) \quad \int_S h(s)[v_N - c]ds = 0, \quad \forall h \in \mathcal{H}.$$

We will need the following lemma:

Lemma A. *The set of restrictions on S of all harmonic functions in D is dense in $L^2(S)$.*

Proof of Lemma A. We give a proof for the convenience of the reader.

The proof is borrowed from [12]. Suppose that $g \in L^2(S)$, and $\int_S g(s)h(s)ds = 0 \quad \forall h \in \mathcal{H}$. Since $(4\pi|x - y|)^{-1}$ is in \mathcal{H} if $y \in D'$, one gets

$$w(y) := \int_S g(s)(4\pi|s - y|)^{-1}ds = 0 \quad \forall y \in D'.$$

Thus, a single layer potential w , with L^2 density g , vanishes in D' , and, by continuity, on S . Since w is a harmonic function in D vanishing on S , it follows that $w = 0$ in D . By the jump formula for the normal derivative of the single-layer potential across a Lipschitz boundary, one gets $g = 0$. \square

Thus, (8) implies $v_N|_S = c$. Therefore, (4) holds.

A result, related to equation (5), was studied in [7] for a two-dimensional problem. The arguments in [7] were not quite clear to the author.

Our main result is a new proof of Theorem 1. The proof is simple, and the method of the proof is new. This method can be used in other problems (see [5], [10],[12], [11]).

2 Proofs

Proof of Theorem 1. We denote by D' the complement of D in \mathbb{R}^3 , by S^2 the unit sphere, by $[a, b]$ the cross product of two vectors, by $g = g(\phi)$ the rotation about an axis, directed along a vector $\alpha \in S^2$, by the angle ϕ , and note that if $h(x)$ is a harmonic function in any ball B_R , containing D , then $h(gx)$ is also a harmonic function in B_R .

Take $h = h(g(\phi)x)$ in (5), differentiate with respect to ϕ and then set $\phi = 0$. This yields:

$$\int_D \nabla h(x) \cdot [\alpha, x] dx = c \int_S \nabla h(s) \cdot [\alpha, s] ds.$$

Using the divergence theorem, one rewrites this as:

$$\alpha \cdot \int_S [s, N] h(s) ds = \alpha \cdot \int_S [s, c \nabla h(s)] ds.$$

Since $\alpha \in S^2$ is arbitrary, one gets

$$(9) \quad \int_S [s, N] h(s) ds = \int_S [s, c \nabla h(s)] ds, \quad \forall h \in \mathcal{H},$$

where $N = N_s$ is a unit normal to S at the point $s \in S$, pointing into D' .

Let $y \in B'_R$ be an arbitrary point, and $h(x) = \frac{1}{|x-y|} \in \mathcal{H}$, where $x \in B_R$. Then equation (9) implies that

$$(10) \quad v(y) := \int_S \frac{[s, N] ds}{|s-y|} = c \left[\nabla \int_S \frac{ds}{|s-y|}, y \right], \quad \forall y \in B'_R,$$

because

$$(11) \quad c \int_S [s, \nabla_s \frac{1}{|s-y|}] ds = c \int_S [\frac{s}{|s-y|^3}, y] ds = c [\nabla_y \int_S \frac{ds}{|s-y|}, y].$$

Relation (11) actually holds for all $y \in D'$, because of the analyticity of its left and right sides in D' . Let

$$w(y) := \int_S |s - y|^{-1} ds.$$

Denote $y^0 := y/|y|$. It is known (see, e.g., [3]) that

$$(12) \quad |y-s|^{-1} = \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{4\pi}{2n+1} Y_{nm}(y^0) \overline{Y_{nm}(s^0)} |s|^n |y|^{-(n+1)}, \quad |y| > |s|,$$

where the overline stands for the complex conjugate, y^0 is the unit vector characterized by the angles θ, ϕ in spherical coordinates, Y_{nm} are normalized spherical harmonics:

$$Y_{nm}(y^0) = Y_{nm}(\theta, \phi) = \gamma_{nm} P_{n,|m|}(\cos \theta) e^{im\phi}, \quad -n \leq m \leq n,$$

$\gamma_{nm} = [\frac{(2n+1)(n-m)!}{4\pi(n+m)!}]^{1/2}$ are normalizing constants:

$$(Y_{nm}(y^0), Y_{pq}(y^0))_{L^2(S^2)} = \delta_{np} \delta_{mq},$$

and

$$P_{n,|m|}(\cos \theta) = (\sin \theta)^{|m|} \left(\frac{d}{d \cos \theta} \right)^{|m|} P_n(\cos \theta)$$

are the associated Legendre functions, where $P_n(\cos \theta)$ are the Legendre polynomials.

If $z = \cos \theta$, then

$$P_{n,m}(z) = (z^2 - 1)^{m/2} \left(\frac{d}{dz} \right)^m P_n(z), \quad m = 1, 2, \dots,$$

$$P_n(z) = (2^n n!)^{-1} \left(\frac{d}{dz} \right)^n (z^2 - 1)^n, \quad P_0(z) = 1$$

(see [3]). The definitions of $P_{n,m}(z)$ in various books can differ by a factor $(-1)^m$.

Using formula (12), one obtains

$$(13) \quad w(y) = \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \sum_{m=-n}^n Y_{nm}(y^0) |y|^{-(n+1)} c_{nm}, \quad c_{nm} := \int_S |s|^n \overline{Y_{nm}(s^0)} ds.$$

Substitute this in (10), equate the terms in front of $|y|^{-(n+1)}$, and define vectors

$$(14) \quad a_{nm} := \int_S [s, N] |s|^n \overline{Y_{nm}(s^0)} ds$$

to obtain

$$(15) \quad \sum_{m=-n}^n Y_{nm}(y^0) a_{nm} = \sum_{m=-n}^n cc_{nm} [e_\theta \partial_\theta Y_{nm}(y^0) + e_\phi (\sin \theta)^{-1} \partial_\phi Y_{nm}(y^0), e_r],$$

where e_θ, e_ϕ , and e_r are orthogonal unit vectors of the spherical coordinate system, $[e_\phi, e_r]$ is the cross product, $[e_\phi, e_r] = e_\theta$, $[e_\theta, e_r] = -e_\phi$, $y = ry^0$, $r = |y|$, $y^0 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, $\partial_\theta = \frac{\partial}{\partial \theta}$.

Therefore, formula (15) can be rewritten as

$$(16) \quad \sum_{m=-n}^n Y_{nm}(y^0) a_{nm} = \sum_{m=-n}^n cc_{nm} \left(-e_\phi \partial_\theta Y_{nm}(y^0) + e_\theta (\sin \theta)^{-1} \partial_\phi Y_{nm}(y^0) \right).$$

From (16) we want to derive that

$$(17) \quad a_{nm} = 0, \quad n \geq 0, \quad -n \leq m \leq n.$$

If (17) is established, then it follows from (14) and from the completeness in $L^2(S)$ of the system $\{|s|^n Y_{nm}(s^0)\}_{n \geq 0, -n \leq m \leq n}$ that $[s, N] = 0$ on S , and this implies that S is a sphere, as follows from Lemma 1 formulated and proved below. Consequently, Theorem 1 is proved as soon as relations (17) are established. The completeness of the system $\{|s|^n Y_{nm}(s^0)\}_{n \geq 0, -n \leq m \leq n}$ in $L^2(S)$ follows from Lemma B:

The functions $|x|^n Y_{nm}(x^0)$, $n \geq 0$, $-n \leq m \leq n$, are harmonic in any ball, centered at the origin.

Lemma B. *The set of restrictions of the above functions to any Lipschitz surface homeomorphic to a sphere is complete in $L^2(S)$.*

Proof of Lemma B. The proof is given for completeness. It is similar to the proof of Lemma A. Suppose that $g \in L^2(S)$ and

$$\int_S g(s) |s|^n Y_{nm}(s^0) ds = 0, \quad \forall n \geq 0, |m| \leq n.$$

This and (12) imply that

$$\int_S g(s)(4\pi|s-y|)^{-1}ds = 0 \quad \forall y \in D',$$

and the argument, given in the proof of Lemma A, yields the desired conclusion $g = 0$. \square

Vector a_{nm} is written in Cartesian basis $\{e_j\}_{1 \leq j \leq 3}$ as

$$a_{nm} = \sum_{j=1}^3 a_{nm,j} e_j.$$

The relation between the components F_1, F_2, F_3 , of a vector F in Cartesian coordinates and its components F_r, F_θ, F_ϕ , in spherical coordinates can be found, e.g., in [6], Section 6.5:

$$F_1 = F_r \sin \theta \cos \phi + F_\theta \cos \theta \cos \phi - F_\phi \sin \phi,$$

$$F_2 = F_r \sin \theta \sin \phi + F_\theta \cos \theta \sin \phi + F_\phi \cos \phi,$$

$$F_3 = F_r \cos \theta - F_\theta \sin \theta.$$

Using these relations one derives from (16) the following formulas:

$$(18) \quad \sum_{m=-n}^n a_{nm,1} Y_{nm}(y^0) = \sum_{m=-n}^n c c_{nm} \left(\partial_\theta Y_{nm}(y^0) \sin \phi + \partial_\phi Y_{nm}(y^0) \cot \theta \cos \phi \right),$$

$$(19) \quad \sum_{m=-n}^n a_{nm,2} Y_{nm}(y^0) = \sum_{m=-n}^n c c_{nm} \left(-\partial_\theta Y_{nm}(y^0) \cos \phi + \partial_\phi Y_{nm}(y^0) \cot \theta \sin \phi \right),$$

$$(20) \quad \sum_{m=-n}^n a_{nm,3} Y_{nm}(y^0) = - \sum_{m=-n}^n c c_{nm} \partial_\phi Y_{nm}(y^0).$$

From formulas (18)-(20) one derives (17).

If $n = 0$, then $a_{00} = 0$, as the following calculation shows:

$$a_{00} = \frac{1}{(4\pi)^{1/2}} \int_S [s, N] ds = -\frac{1}{(4\pi)^{1/2}} \int_D [\nabla, x] dx = 0.$$

If $n > 0$, then multiply equation (20) by $e^{-im\phi}$, integrate with respect to ϕ over $[0, 2\pi]$, write $P_{n,m}$ for $P_{n,m}(\cos \theta)$, and obtain

$$(21) \quad a_{nm,3}P_{n,m} = -cc_{n,m}imP_{n,m}, \quad c_{n,m} := c_{nm}.$$

One concludes that $a_{n0,3} = 0$ and $a_{nm,3} = -imcc_{n,m}$.

If one derives from (18)-(19) that $c_{n,m} = 0$, then equation (17) follows, and Theorem 1 is proved.

From (18) and (19) one derives analogs of (21):

$$(22) \quad \begin{aligned} 2ia_{nm,1}\gamma_{nm}P_{n,m} &= cc_{n,m-1}\gamma_{n,m-1}(\partial_\theta P_{n,m-1} - (m-1)\cot\theta P_{n,m-1}) \\ &\quad - cc_{n,m+1}\gamma_{n,m+1}(\partial_\theta P_{n,m+1} + (m+1)\cot\theta P_{n,m+1}), \end{aligned}$$

$$(23) \quad \begin{aligned} 2a_{nm,2}\gamma_{nm}P_{n,m} &= cc_{n,m-1}\gamma_{n,m-1}(-\partial_\theta P_{n,m-1} + (m-1)\cot\theta P_{n,m-1}) \\ &\quad - cc_{n,m+1}\gamma_{n,m+1}(\partial_\theta P_{n,m+1} + (m+1)\cot\theta P_{n,m+1}). \end{aligned}$$

Let us take $\theta \rightarrow 0$ in the above equations. It is known (see [3], Section 3.9.2, formula (4)) that

$$(24) \quad P_{n,m}(z) \sim b(n,m)(z-1)^{m/2}, \quad z \rightarrow 1, \quad b(n,m) := \frac{(n+m)!}{2^{m/2}m!(n-m)!}.$$

Equation (22) can be considered as a linear combination

$$(25) \quad \sum_{j=1}^3 A_j f_j(z) = 0,$$

where the A_j are constants:

$$A_1 = 2ia_{nm,1}\gamma_{nm}, \quad A_2 = -cc_{n,m-1}\gamma_{n,m-1}, \quad A_3 = cc_{n,m+1}\gamma_{n,m+1},$$

and

$$\begin{aligned} f_1(z) &= P_{n,m}(z), \\ f_2(z) &= -(1-z^2)^{1/2}P'_{n,m-1}(z) - (m-1)\frac{z}{(1-z^2)^{1/2}}P_{n,m-1}(z), \\ f_3(z) &= -(1-z^2)^{1/2}P'_{n,m+1}(z) - (m+1)\frac{z}{(1-z^2)^{1/2}}P_{n,m+1}(z), \quad z = \cos \theta. \end{aligned}$$

If the system of functions $\{f_j(z)\}_{j=1}^3$ is linearly independent on the interval $[-1, 1]$, then all $A_j = 0$ in (25), that is, $A_1 = 0$, $A_2 = 0$, and $A_3 = 0$. This implies that

$$a_{nm,1} = c_{n,m} = 0, \quad -n \leq m \leq n.$$

The quantities $a_{nm,2}$ and $a_{nm,3}$ are proportional to $c_{n,m}$. Since $c_{n,m} = 0$, it follows that

$$a_{nm,2} = a_{nm,3} = 0, \quad -n \leq m \leq n,$$

and Theorem 1 is proved.

Thus, to complete the proof of Theorem 1 it is sufficient to verify the linear independence of the system of functions $\{f_j(z)\}_{j=1}^3$ on the interval $z \in [-1, 1]$.

From formula (24) it follows that these functions have the following main terms of their asymptotics as $z \rightarrow 1$:

$$f_1(z) \sim B_1(z-1)^{m/2}, \quad f_2(z) \sim B_2 \frac{(z-1)^{(m+1)/2}}{(1-z^2)^{1/2}}, \quad f_3(z) \sim B_3 \frac{(z-1)^{(m+3)/2}}{(1-z^2)^{1/2}},$$

where the constants $B_j \neq 0$, $1 \leq j \leq 3$, depend on n, m . The linear independence of the system $\{f_j(z)\}_{j=1}^3$ holds because the system

$$\{(z-1)^{m/2}, \quad \frac{(z-1)^{(m+1)/2}}{(1-z^2)^{1/2}}, \quad \frac{(z-1)^{(m+3)/2}}{(1-z^2)^{1/2}}\}$$

is linearly independent. The linear independence of this system holds if the system

$$\{1, \quad (1+z)^{-0.5}, \quad (z-1)(1+z)^{-0.5}\}$$

is linearly independent on the interval $[-1, 1]$. The linear independence of this system on the interval $[-1, 1]$ is obvious.

Theorem 1 is proved. \square

Lemma 1. *If S is a C^2 -smooth closed surface and $[s, N_s] = 0$ on S , then S is a sphere.*

Proof of Lemma 1. Let $s = r(u, v)$ be a parametric equation of S . Then the vectors r_u and r_v are linearly independent and N_s is directed along the vector $[r_u, r_v]$. Thus, the assumption $[s, N_s] = 0$ on S implies that

$$[r, [r_u, r_v]] = r_u(r, r_v) - r_v(r, r_u) = 0.$$

Since the vectors r_u and r_v are linearly independent, it follows that $(r, r_v) = (r, r_u) = 0$. Thus, $(r, r) = R^2$, where R^2 is a constant. This means that S is a sphere. Lemma 1 is proved. \square

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